Riemannian Center of Mass

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The Riemannian center of mass (RCM) [Karcher, 2014] is a generalization of the center of mass to manifolds. In Euclidean space, the center of mass is the unique point of a kinematic system where, if we would support the system at this point, the system would be perfectly balanced. In euclidean space with N points, this can be formulated using the weighted arithmetic mean as

$$x_{\mathrm{CM}} = \frac{\sum\limits_{i=1}^{N} w_i \cdot x_i}{\sum\limits_{i=1}^{N} w_i} \tag{1}$$

whereby x_{CM} is the center of mass (CM) for points $\{x_1, \cdots, x_N\} \in \mathbb{R}^m$ and w_i are non-negative weights. We assume in the following without loss of generality (w.l.o.g.) that $\sum\limits_{i=1}^{N} w_i = 1$, which can be thought of as a rescaling of the weights. This does not change anything in the equations.

1 Vector Field Formulation

When the points do not lie in Euclidean space, but on a manifold M, the case becomes more complicated. Computing the CM would result in a CM outside the manifold M, which does not represent a valid point. A way to deal with this case was presented by german mathematician Hermann Karcher [Karcher, 1977, 2014]. His idea was to formulate the Euclidean center of mass as a vector field on Euclidean space, which he defined as

$$V(x) = \sum_{i=1}^{N} w_i \cdot (x_i - x). \tag{2}$$

This represents for any point $x \in \mathbb{R}^m$ a direction which points towards the center of mass. This can be seen by realizing that (a) the vector field vanishes at x_{CM}, and that (b) if x is a random point, then the direction vector from V at x equals $x_{\mathrm{CM}} - x$, i.e. it points towards x_{CM}. Both of those claims are proven in Sec. 2.1.

2 Vector Field on Manifolds

It turns out that such a vector field formulation can be transferred to an arbitrary manifold M. Karcher Karcher [2014] does it in the following way: He notes that $V(x)$ represents a tangent vector on the Euclidean space along a straight line towards the center of mass. Since M is probably curved, we would $V(x)$ like to represent a tangent vector *along the geodesic* from x to x_{CM}.

Finding the mean then amounts to following the gradient of this vector field. A computationally efficient way is to compute the RCM incrementally [Salehian, 2014][Ch. 3]. The idea is simple and analogues to the euclidean case: If we start with two points, the mean lies in the middle of the geodesic (shortest path) between those two points. Once we add a third point, we need to move the mean towards the third point. How much do you ask? Well, its influence is exactly one third, because the mean accounts for two thirds since it was created by two points. On the manifold, you do the same, but now we move along geodesics.

In general, if an N-th point $x_N \in M$ is added, we move a distance of $\frac{1}{N}$ along the geodesic from the previous mean point m_{N-1} to the new x_N point. The incremental update rule for the mean is then

$$m_N = m_{N-1} + \gamma\left(\frac{1}{N}\right) \tag{3}$$

whereby $\gamma : [0,1] \to M$ is the geodesic on M from $\gamma(0) = m_{N-1}$ to $\gamma(1) = x_N$ and $m_1 = x_1$. Note that the RCM is a local property, i.e. it might not unique. Think about two points directly opposite on a circle—two possible solutions are valid center of masses.

2.1 Proofs

2.1.1 Vector field vanishes

To see that the vector field vanishes at x_{CM}, we can put it into equation (2). For convenience and without loss of generality, we assume that $\sum_{i=1}^{N} w_i = 1$.

$$
\begin{aligned}
V(x) &= \sum_{i=1}^{N} w_i \cdot (x_i - x_{\text{CM}}) \\
&= \sum_{i=1}^{N} w_i \cdot (x_i - \sum_{k=1}^{N} w_k \cdot x_k) \\
&= \sum_{i=1}^{N} w_i x_i - \sum_{i=1}^{N} w_i \sum_{k=1}^{N} w_k x_k \\
&= \sum_{i=1}^{N} w_i x_i (1 - \sum_{i=1}^{N} w_i) \\
&= \sum_{i=1}^{N} w_i x_i (1 - 1) \\
&= 0
\end{aligned}
$$

2.1.2 Vector field points towards center of mass

To see that the vector field points towards x_{CM} at any point x, we can put it into equation (2).

$$
\begin{aligned}
V(x) &= \sum_{i=1}^{N} w_i \cdot (x_i - x) \\
&= \sum_{i=1}^{N} w_i x_i - \sum_{i=1}^{N} w_i x \\
&= x_{\text{CM}} - \sum_{i=1}^{N} w_i x \\
&= x_{\text{CM}} - x
\end{aligned}
$$

References

Hermann Karcher. Riemannian center of mass and mollifier smoothing. *Communications on pure and applied mathematics*, 30(5):509–541, 1977.

Hermann Karcher. Riemannian center of mass and so called karcher mean. *arXiv preprint arXiv:1407.2087*, 2014.

Hesamoddin Salehian. *Incremental algorithms for statistical analysis of manifold valued data*. University of Florida, 2014.