Convexity Constraints in Optimization over Functional Spaces

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Let $\bar{\tau} \in \mathcal{F} = C([0,1], X)$ with $\mathcal{F}$ being an infinite dimensional functional space containing all continuous functions from $[0,1]$ to a space X [Kreyszig, 1991]. We can represent $\mathcal{F}$ by an orthonormal basis of an infinite number of basis functions $\{f_1, f_2, \cdots\}$. Any function $\bar{\tau}$ in $\mathcal{F}$ can then be written as a linear combination of basis functions as [Boyd and Vandenberghe, 2004]

$$\bar{\tau}(t) = \sum_{k=0}^{\infty} w_k f_k(t) \tag{1}$$

whereby w_k are called the coefficients of the basis functions. Optimization over $\mathcal{F}$ can now be done by optimizing over the linear coefficients w_k, corresponding to a linear program.

$$\underset{\{w_1, w_2, \cdots\}}{\text{optimize}} \quad \bar{\tau} = \sum_{k=0}^{\infty} w_k f_k(t) \tag{2}$$

Let us approximate this linear program by using only a finite number of basis function, corresponding to a subspace of $\mathcal{F}$. This corresponds to a loss of completeness. However, for applications where for example high frequency functions are undesirable, this approximation will actually be almost not noticeable. Let us choose a $K \gg 0$ such that

$$\underset{\{w_1, \cdots, w_K\}}{\text{optimize}} \quad \tau = \sum_{k=0}^{K} w_k f_k(t) \tag{3}$$

This convex (linear) program will be our (approximate) representation of the functional space $\mathcal{F}$.

1 Classical Convex Constraints

We can now apply any convex constraints on the functional space, so that we preserve convexity of our optimization procedure. A good overview is given by Boyd and Vandenberghe [2004]. We will try to list here all known convex constraints, whereby

- LEC = Linear Equality Constraint
- LIC = Linear Inequality Constraint
- QEC = Quadratic Equality Constraint
- QIC = Quadratic Inequality Constraint
- CEC = Convex Equality Constraint
- CIC = Convex Inequality Constraint

1.1 Interpolation Condition (LEC)

At point t_i, we like the function τ to have the value z_i, corresponding for example to a waypoint at z_i.

$$\boxed{\tau(t_i) = z_i, \quad i = 1, \cdots, m} \tag{4}$$

this is an LEC on the functional space.

1.2 Neighborhood Constraint (LIC)

We can also require the function to be in an ϵ-neighborhood of the waypoint at point t_i as:

$$\boxed{\|\tau(t_i) - z_i\|_2 \leq \epsilon \quad i = 1, \cdots, m} \tag{5}$$

1.3 Polytope Constraint (LIC)

Going even further, we can restrict the position of τ at point t_i to any convex polytope $P = \{x | Ax \leq b\}$ as

$$\boxed{A\tau(t_i) \leq b \quad i = 1, \cdots, m} \tag{6}$$

1.4 Lipschitz Constraint (LIC)

$$\boxed{\|\tau(t_j) - \tau(t_k)\| \leq L\|t_j - t_k\| \quad i = 1, \cdots, m} \tag{7}$$

2 Convex Derivative Constraints

Let us assume that $\mathcal{F}$ is actually $\mathcal{C}^1$, contains only functions which are differentiable. Given a point t_i, the derivative of τ is given by

$$\frac{d}{dt}\tau(t) = \sum_{k=0}^{K} w_k \frac{d}{dt} f_k(t) \tag{8}$$

which is again a linear function of x. We can therefore apply any constraints which we applied to τ also to its derivative $\dfrac{d}{dt}\tau$.

2.1 Maximum Gradient Norm (QIC)

If we like to constraint the maximum gradient of τ (corresponding to a maximum speed), we can do this via a QIC

$$\boxed{\|\frac{d}{dt}\tau(t_i)\| \leq M} \tag{9}$$

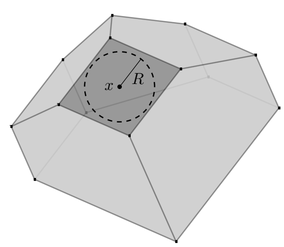
2.2 Monotonity (LIC)

If we like our function to be increasing we can restrict the derivative to the positive halfspace

$$\boxed{\tau(t_i) \geq 0 \quad i = 1, \cdots, m} \tag{10}$$

3 Combination Constraints

3.1 Circle on Polygonal Surface Constraint



Let a polytope be given by $P = \{x \in R^n | a_k x \leq b_k, k = 1, \cdots, D\}$, non-empty, bounded. Given a surface p of this polytope, we can construct a convex equation to determine if a point on the surface p has a circle of radius R, which is fully contained in the surface element p.

$$\begin{aligned} a_k^T x + R a_k^T a_k' \leq b_k, \quad i = \{1, \cdots, p-1, p+1, \cdots, T\} \\ a_p^T x = b_p \end{aligned} \tag{11}$$

whereby R is the radius of the circle, x the center, a_p is the surface normal of the element p, and a'_k is the orthogonal projection onto the hyperplane of a_p, i.e. $a'_k = a_k - (a_k^T a_p)a_p$. This could correspond to the (circular) foot of a humanoid robot stepping on a box or another surface element. If our trajectory is actually a trajectory for a foot, we might want to restrict the whole foot to this surface element. The corresponding constraint on our trajectory would be

$$\boxed{A\tau(t_i) \leq b - \epsilon\,\mathbf{diag}(A^T A')} \tag{12}$$

whereby we have $A = \{a_1, \cdots, a_D\}$ and $A' = \{a'_1, \cdots, a'_D\}$ with $a'_k = a_k - (a_k^T a_p)a_p$

References

Stephen Boyd and Lieven Vandenberghe. *Convex Optimization.* Cambridge university press, 2004.

Erwin Kreyszig. *Introductory functional analysis with applications.* John Wiley & Sons, 1991.