Riemannian Center of Mass

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The Riemannian center of mass (RCM) [Karcher, 2014] is a generalization of the center of mass to manifolds. In Euclidean space, the center of mass is the unique point of a kinematic system where, if we would support the system at this point, the system would be perfectly balanced. In Euclidean space with N points, this can be formulated using the weighted arithmetic mean as

$$x_{\mathrm{CM}} = \sum_{i=1}^{N} w_i \cdot x_i \tag{1}$$

whereby x_{CM} is the center of mass (CM) for points $\{x_1, \cdots, x_N\} \in \mathbb{R}^m$ and w_i are non-negative weights such that $\sum_{i=1}^{N} w_i = 1$. Note that any weights are possible, and can be represented as Eq. 1 by a rescaling of the weights.

1 Vector Field Formulation

When the points do not lie in Euclidean space, but on a manifold M, the case becomes more complicated. Computing the CM on a manifold M would almost always result in a CM outside the manifold M, which does not represent a valid point. Imagine computing the center of mass for points on a circle—the resulting CM would almost always lie away from the circle, not on the circle. A way to deal with such cases was presented by german mathematician Hermann Karcher [Karcher, 1977, 2014]. His idea was to reformulate the Euclidean center of mass as a vector field on Euclidean space, which he defined as

$$V(x) = \sum_{i=1}^{N} w_i \cdot (x_i - x). \tag{2}$$

This represents for any point $x \in \mathbb{R}^m$ a direction which points towards the center of mass. This can be seen by realizing that (a) the vector field vanishes at x_{CM}, and that (b) if x is a random point, then the direction vector from V at x equals $x_{\mathrm{CM}} - x$, i.e. it points towards x_{CM}. Both of those claims are proven in Sec. 2.1. Given this vector field, the mean can be found by following the gradient until the vector field vanishes.

2 Vector Field on Manifolds

It turns out that such a vector field formulation can be transferred to an arbitrary manifold M. [Karcher, 2014] does it in the following way: He notes that $V(x)$ represents a tangent vector on the Euclidean space along a straight line towards the center of mass. Since M is probably curved, we would $V(x)$ like to represent a tangent vector *along the geodesic* from x to x_{CM}. Note that this center of mass lies exclusively inside the manifold. It is therefore called the Riemannian center of mass (RCM).

Finding the RCM then amounts to following the gradient of this vector field on the manifold. A computationally efficient way is to compute the RCM incrementally [Salehian, 2014][Ch. 3]. The idea is simple and analogues to the euclidean case. Let us assume that we start with two points and all points are equally weighted. The center of mass lies in the middle of the geodesic (shortest path) between those two points. Once we add a third point, we need to move the center of mass towards the third point. This can be accomplished by imagining that the center of mass has the weight of the first two points. To find the new center of mass, we move one third in the direction of the third point.

In general, if a point $x_N \in M$ is added to an existing RCM m_{N-1}, we need to move a distance of $\frac{1}{N}$ along the geodesic from the previous mean m_{N-1} to the new x_N point to reach the new RCM m_N. The incremental update rule for the mean is then

$$m_N = m_{N-1} + \gamma\left(\frac{1}{N}\right) \tag{3}$$

whereby $\gamma : [0,1] \to M$ is the geodesic on M from $\gamma(0) = m_{N-1}$ to $\gamma(1) = x_N$ and $m_1 = x_1$. Note that the RCM is a local property, i.e. it might not be unique. Think about two points directly opposite on a circle—two possible solutions are valid center of masses.

2.1 Proofs

Here we provide the two proofs that the vector field $V(x)$ vanishes at the center of mass, and that the vector field always points towards the center of mass.

2.1.1 Vector Field Vanishes at Center of Mass

To see that the vector field vanishes at x_{CM}, we can insert the definition of x_{CM} from Eq. 1 into Eq. (2) and compute the result. For convenience and without loss of generality, we assume that $\sum_{i=1}^{N} w_i = 1$. This results in

$$\begin{aligned}
V(x_{\mathrm{CM}}) &= \sum_{i=1}^{N} w_i \cdot (x_i - x_{\mathrm{CM}}) \\
&= \sum_{i=1}^{N} w_i \cdot (x_i - \sum_{k=1}^{N} w_k \cdot x_k) \\
&= \sum_{i=1}^{N} w_i x_i - \sum_{i=1}^{N} w_i \sum_{k=1}^{N} w_k x_k \\
&= \sum_{i=1}^{N} w_i x_i (1 - \sum_{i=1}^{N} w_i) \\
&= \sum_{i=1}^{N} w_i x_i (1 - 1) \\
&= 0,
\end{aligned}$$

which shows that the vector field vanishes at x_{CM}.

2.1.2 Vector Field Points Towards Center of Mass

To see that this vector field points towards x_{CM} at any point x, we can rearrange Eq. (2) to give an expression for x_{CM} as

$$\begin{aligned}
V(x) &= \sum_{i=1}^{N} w_i \cdot (x_i - x) \\
&= \sum_{i=1}^{N} w_i x_i - \sum_{i=1}^{N} w_i x \\
&= x_{\mathrm{CM}} - \sum_{i=1}^{N} w_i x \\
&= x_{\mathrm{CM}} - x.
\end{aligned}$$

References

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