

# Convexity Constraints in Optimization over Functional Spaces

Andreas Orthey

April 2014

Let  $\bar{\tau} \in \mathcal{F} = C([0, 1], X)$  with  $\mathcal{F}$  being an infinite dimensional functional space containing all continuous functions from  $[0, 1]$  to a space  $X$  [?]. We can represent  $\mathcal{F}$  by an orthonormal basis of an infinite number of basis functions  $\{f_1, f_2, \dots\}$ . Any function  $\bar{\tau}$  in  $\mathcal{F}$  can then be written as a linear combination of basis functions as [?]

$$\bar{\tau}(t) = \sum_{k=0}^{\infty} w_k f_k(t) \quad (1)$$

whereby  $w_k$  are called the coefficients of the basis functions. Optimization over  $\mathcal{F}$  can now be done by optimizing over the linear coefficients  $w_k$ , corresponding to a linear program.

$$\text{optimize}_{\{w_1, w_2, \dots\}} \quad \bar{\tau} = \sum_{k=0}^{\infty} w_k f_k(t) \quad (2)$$

Let us approximate this linear program by using only a finite number of basis function, corresponding to a subspace of  $\mathcal{F}$ . This corresponds to a loss of completeness. However, for applications where for example high frequency functions are undesirable, this approximation will actually be almost not noticeable. Let us choose a  $K \gg 0$  such that

$$\text{optimize}_{\{w_1, \dots, w_K\}} \quad \tau = \sum_{k=0}^K w_k f_k(t) \quad (3)$$

This convex (linear) program will be our (approximate) representation of the functional space  $\mathcal{F}$ .

# 1 Classical Convex Constraints

We can now apply any convex constraints on the functional space, so that we preserve convexity of our optimization procedure. A good overview is given by ?. We will try to list here all known convex constraints, whereby

- LEC = Linear Equality Constraint
- LIC = Linear Inequality Constraint
- QEC = Quadratic Equality Constraint
- QIC = Quadratic Inequality Constraint
- CEC = Convex Equality Constraint
- CIC = Convex Inequality Constraint

## 1.1 Interpolation Condition (LEC)

At point  $t_i$ , we like the function  $\tau$  to have the value  $z_i$ , corresponding for example to a waypoint at  $z_i$ .

$$\tau(t_i) = z_i, \quad i = 1, \dots, m \quad (4)$$

this is an LEC on the functional space.

## 1.2 Neighborhood Constraint (LIC)

We can also require the function to be in an  $\epsilon$ -neighborhood of the waypoint at point  $t_i$  as:

$$\|\tau(t_i) - z_i\|_2 \leq \epsilon \quad i = 1, \dots, m \quad (5)$$

## 1.3 Polytope Constraint (LIC)

Going even further, we can restrict the position of  $\tau$  at point  $t_i$  to any convex polytope  $P = \{x | Ax \leq b\}$  as

$$A\tau(t_i) \leq b \quad i = 1, \dots, m \quad (6)$$

## 1.4 Lipschitz Constraint (LIC)

$$\|\tau(t_j) - \tau(t_k)\| \leq L\|t_j - t_k\| \quad i = 1, \dots, m \quad (7)$$

## 2 Convex Derivative Constraints

Let us assume that  $\mathcal{F}$  is actually  $\mathcal{C}^1$ , contains only functions which are differentiable. Given a point  $t_i$ , the derivative of  $\tau$  is given by

$$\frac{d}{dt}\tau(t) = \sum_{k=0}^K w_k \frac{d}{dt}f_k(t) \quad (8)$$

which is again a linear function of  $x$ . We can therefore apply any constraints which we applied to  $\tau$  also to its derivative  $\frac{d}{dt}\tau$ .

### 2.1 Maximum Gradient Norm (QIC)

If we like to constraint the maximum gradient of  $\tau$  (corresponding to a maximum speed), we can do this via a QIC

$$\boxed{\left\| \frac{d}{dt}\tau(t_i) \right\| \leq M} \quad (9)$$

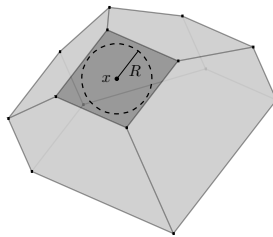
### 2.2 Monotony (LIC)

If we like our function to be increasing we can restrict the derivative to the positive halfspace

$$\boxed{\tau(t_i) \geq 0 \quad i = 1, \dots, m} \quad (10)$$

## 3 Combination Constraints

### 3.1 Circle on Polygonal Surface Constraint



Let a polytope be given by  $P = \{x \in R^n | a_k x \leq b_k, k = 1, \dots, D\}$ , non-empty, bounded. Given a surface  $p$  of this polytope, we can construct a convex equation to determine if a point on the surface  $p$  has a circle of radius  $R$ , which is fully contained in the surface element  $p$ .

$$\begin{aligned} a_k^T x + R a_k^T a'_k &\leq b_k, \quad i = \{1, \dots, p-1, p+1, \dots, T\} \\ a_p^T x &= b_p \end{aligned} \quad (11)$$

whereby  $R$  is the radius of the circle,  $x$  the center,  $a_p$  is the surface normal of the element  $p$ , and  $a'_k$  is the orthogonal projection onto the hyperplane of  $a_p$ , i.e.  $a'_k = a_k - (a_k^T a_p) a_p$ . This could correspond to the (circular) foot of a humanoid robot stepping on a box or another surface element. If our trajectory is actually a trajectory for a foot, we might want to restrict the whole foot to this surface element. The corresponding constraint on our trajectory would be

$$\boxed{A\tau(t_i) \leq b - \epsilon \mathbf{diag}(A^T A')} \quad (12)$$

whereby we have  $A = \{a_1, \dots, a_D\}$  and  $A' = \{a'_1, \dots, a'_D\}$  with  $a'_k = a_k - (a_k^T a_p) a_p$